

ON INTERSECTION FORMS OF DEFINITE 4-MANIFOLDS BOUNDED BY A RATIONAL HOMOLOGY 3-SPHERE

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ABSTRACT. We show that, if a rational homology 3-sphere Y bounds a positive definite smooth 4-manifold, then there are finitely many negative definite lattices, up to the stable-equivalence, which can be realized as the intersection form of a smooth 4-manifold bounded by Y . To this end, we make use of constraints on definite forms bounded by Y induced from Donaldson's diagonalization theorem, and Ozsváth and Szabó's Heegaard Floer correction term. We also present some families of Seifert fibered 3-manifolds that bound both positive and negative definite smooth 4-manifolds.

1. INTRODUCTION

Throughout this paper we assume that all manifolds are compact and oriented. We say a 4-manifold X is *bounded by* a 3-manifold Y if Y is homeomorphic to the boundary of X and the orientation of Y inherits the orientation of X in the standard way.

The intersection pairing Q_X on $H_2(X; \mathbb{Z})/Tors$ of a 4-manifold X is an integer-valued, symmetric, bilinear form over \mathbb{Z}^n . If the boundary of X is a rational homology 3-sphere or empty, then Q_X is nondegenerate. In this case, Q_X algebraically forms a lattice. In particular, if X is closed, then Q_X is unimodular, i.e. $\det(Q_X) = \pm 1$.

The remarkable works of Donaldson and Freedman in the early 1980's portray a big difference between topological and smooth categories in dimension 4 for the answer to the following question:

Question. Which negative definite, unimodular lattices are realized as the intersection form of a closed 4-manifold?

Freedman showed that any unimodular definite lattice can be realized as the intersection form of a closed, *topological* 4-manifold [Fre82]. On the other hand, Donaldson's diagonalization theorem asserts that the only standard one can be realized as the intersection form of a closed, *smooth* 4-manifold [Don87].

In this paper, we would like to study this phenomenon for 4-manifolds with a fixed boundary. We say a lattice Λ is *smoothly* (resp. *topologically*) *bounded by* a 3-manifold Y if Λ can be realized as the intersection form of a smooth (resp. topological) 4-manifold bounded by Y . It follows easily by connected summing $\overline{\mathbb{CP}^2}$ to a 4-manifold realizing Λ that if Y bounds a lattice Λ , then it also does $\Lambda \oplus \langle -1 \rangle$. We call this procedure the *stabilization* of Λ .

Definition 1.1. Two negative definite lattices Λ_1 and Λ_2 are *stable-equivalent* if $\Lambda_1 \oplus \langle -1 \rangle^m \cong \Lambda_2 \oplus \langle -1 \rangle^n$ for some non-negative integers m and n .

Let $\mathcal{I}(Y)$ (resp. $\mathcal{I}^{TOP}(Y)$) denote the set of all negative definite lattices that can be smoothly (resp. topologically) bounded by Y , up to the stable-equivalence. In terms of these notations, aforementioned Freedman and Donaldson's results can be interpreted as

$$\mathcal{I}^{TOP}(S^3) = \{[\Lambda] \mid \Lambda: \text{any unimodular negative definite lattice}\}$$

$$\text{and } \mathcal{I}(S^3) = \{[\langle -1 \rangle]\}.$$

Following from the Freedman's result, Boyer studied a realization problem for topological 4-manifolds with a fixed boundary Y [Boy86]. Roughly speaking, any forms presenting the linking pairing of Y can be realized. Hence one can easily observe the following:

Theorem 1.2. *Let Y be a rational homology 3-sphere. Then*

$$|\mathcal{I}^{TOP}(Y)| = \infty.$$

Proof. For the linking pairing of a 3-manifold Y , it is known by Edmonds [Edm05, Section 6] that there is a definite form presenting it. This form is realized by a topological 4-manifold W by Boyer's result. Now, we have infinitely many definite 4-manifolds bounded by Y , up to the stabilization, by connected summing W and closed topological 4-manifolds with nonstandard definite intersection forms. \square

Our main result is as follows.

Theorem 1.3. *Let Y be a rational homology 3-sphere. If Y bounds a positive (resp. negative) definite smooth 4-manifold, then there are only finitely many negative (resp. positive) definite lattices, up to the stable-equivalence, which can be realized as the intersection form of a smooth 4-manifold bounded by Y , i.e. $|\mathcal{I}(Y)| < \infty$.*

Remark. We say a 4-manifold X is negative (resp. positive) definite if $b_2(X) = b_2^-(X)$ (resp. $b_2(X) = b_2^+(X)$). In particular, a 4-manifold with $b_2 = 0$ is considered to be both positive and negative definite.

In [Boy86, Corollary 0.4], Boyer showed that there are only finitely many homeomorphism types of simply-connected 4-manifolds which have given intersection form and boundary. Our theorem gives a bit of answer to the geography problem of simply-connected, smooth 4-manifolds with a fixed boundary.

Corollary 1.4. *If Y bounds a positive (resp. negative) definite smooth 4-manifold, then there are finitely many homeomorphism types of simply-connected, negative (resp. positive) definite smooth 4-manifolds bounded by Y , up to the stabilization.*

To prove our main theorem, we consider a set of lattices \mathcal{L} defined purely algebraically in terms of invariants of a given 3-manifold Y . This set \mathcal{L} contains negative definite lattices, up to the stable-equivalence, which satisfy the conditions for a definite lattice to be smoothly bounded by Y , induced from Donaldson's diagonalization theorem and Ozsváth and Szabó's Heegaard Floer correction term, together with fundamental obstructions from the algebraic topology. See Section 2.2 for details of these conditions. Then Theorem 1.3 readily follows after we show the finiteness of \mathcal{L} .

Theorem 1.5. *Let Γ be a fixed negative definite lattice, and $C > 0$ and $D \in \mathbb{Z}$ be constants. Define $\mathcal{L}(\Gamma, C, D)$ to be the set of negative definite lattices Λ , up to the stable-equivalence, satisfying the following conditions*

- $\det(\Lambda) = D$,
- $d(\Lambda) \leq C$, and
- $\Gamma \oplus \Lambda$ embeds into the standard negative definite lattice of rank, $rk(\Gamma) + rk(\Lambda)$,

where

$$d(\Lambda) := \frac{n - \min_{\xi \in Char(\Lambda)} |\xi \cdot \xi|}{4},$$

and $Char(\Lambda)$ is the set of characteristic covectors of Λ . Then $\mathcal{L}(\Gamma, C, d)$ is finite.

Our proof of Theorem 1.5 is highly inspired by the work of Owens and Strle in [OS12a], where they studied non-unimodular lattices in terms of the length of characteristic covectors. The key idea of our proof is to improve one of their inequalities on the length of characteristic covectors, enough to give an upper bound on the rank of the lattices in $\mathcal{L}(\Gamma, C, d)$.

Remark. We remark that not all lattice which represent a class in $\mathcal{I}(Y)$ can be realized as the intersection form of a smooth 4-manifold bound by Y . It is well known that the Poincaré homology 3-sphere Σ , oriented as the boundary of the $-E_8$ -plumbed 4-manifold, can be obtained by (-1) -framed surgery along the left-handed trefoil knot. Hence $[\langle -1 \rangle] = [\emptyset] \in \mathcal{I}(\Sigma)$. However, one can prove, using a constraint from the Donaldson's diagonalization theorem, that the empty lattice cannot be bound by Σ . See Example 2.4.

Seifert fibered spaces that bound a positive definite 4-manifold. It is interesting to ask which 3-manifolds satisfy the condition in Theorem 1.3, i.e. which 3-manifolds bound a positive definite smooth 4-manifold. Since many of 3-manifolds, including any Seifert fibered rational homology 3-spheres, bound a definite smooth 4-manifold up to the sign (see Proposition A.2), it is more reasonable to find 3-manifolds which can bound definite smooth 4-manifolds of both signs.

It is well known that any lens space has such property. Note that lens spaces are double branched covers of S^3 along 2-bridge knots. As generalizing lens spaces to this direction, the double branched covers of S^3 along quasi-alternating links are known to bound both signs of definite smooth 4-manifolds (even with trivial first homology) [OS05b, Proof of Lemma 3.6]. Note also that lens spaces can be obtained by Dehn-surgery along the unknot. In [OS12b], Owens and Strle classified 3-manifolds obtained by Dehn-surgery on torus knots which can bound definite smooth 4-manifolds of both signs.

In Appendix A, we consider this question for another class of 3-manifolds, Seifert-fibered rational homology 3-spheres, which also contains lens spaces. We completely determine which links of quotient surface singularities bound definite smooth 4-manifolds of both sign.

Generalizing our result to all rational homology 3-spheres. It is not hard to see that the condition we gave in Theorem 1.3 is not a necessary condition. For example, the Poincaré homology 3-sphere Σ , oriented as before, cannot bound any positive definite smooth 4-manifold, by Donaldson's diagonalization theorem. See Example 2.4. However, we can show that there are only finitely many negative definite lattices bounded by Σ as follows. In the proof of Theorem 1.3, we will use an inequality of Ozsváth and Szabó comparing the d -invariant of a lattice Λ with the Heegaard Floer correction term, also denoted by d , of a bounding 3-manifold

Y :

$$d(\Lambda) \leq d(Y).$$

The correction term of Σ is known as 2. On the other hand, in [Elk95a, Elk95b] Elkies showed that there are only 15 negative definite, unimodular lattices of which $d \leq 2$, up to the stable-equivalence. Therefore, it follows from the Ozsváth and Szabó's inequality that $|\mathcal{I}(\Sigma)|$ is at most 15.

The finiteness on the number of definite unimodular lattices are also known for some small $d \leq 6$ [Gau07, NV03]. Thus we know that $|\mathcal{I}(Y)|$ is finite for any homology 3-sphere Y with $d(Y) \leq 6$. However, it is still unknown, as long as the authors know, if there is a such bound on the number of definite lattices for any $d > 6$. For non-unimodular lattices, Owens and Strle [OS12a] gave an analogous result of Elkies for $d = 0$.

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2. PRELIMINARY

In this section we collect some background materials that will be used to prove our main theorems.

2.1. Lattices. A *lattice* of rank n is a free abelian group \mathbb{Z}^n equipped with an integer-valued, nondegenerate, symmetric, bilinear form,

$$Q: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}.$$

Let $\Lambda = (\mathbb{Z}^n, Q)$ be a lattice. By tensoring Λ with \mathbb{R} , one can extend Q to a symmetric bilinear form over the vector space \mathbb{R}^n . We define the *signature* of Λ to be the signature of Q . We say Λ is *positive* (resp. *negative*) *definite* if the signature of Λ equals to the (resp. negative) rank of Λ .

By fixing a basis $\{v_1, \dots, v_n\}$ for Λ , we can represent Λ by an n by n matrix, $[Q(v_i, v_j)]$. The *determinant* of a lattice Λ , $\det(\Lambda)$, is the determinant of a matrix representation of Λ . In particular, if the determinant of a lattice is ± 1 , or equivalently a corresponding matrix is invertible over \mathbb{Z} , we say the lattice is *unimodular*.

The dual lattice $\Lambda^* := \text{Hom}(\Lambda, \mathbb{Z})$ can be identified with the set of elements in $\xi \in \Lambda \otimes \mathbb{R}$ such that $\xi \cdot w \in \mathbb{Z}$ for any $w \in \Lambda$. We call ξ in Λ^* a *characteristic covector* if $\xi \cdot w \equiv w \cdot w$ modulo 2 for any $w \in \Lambda$. We say a lattice Λ_1 *embeds* into Λ_2 if there is a monomorphism from Λ_1 to Λ_2 preserving bilinear forms. Note that Λ is naturally contained in Λ^* , and the following map is induced by Q :

$$\Lambda^*/\Lambda \otimes \Lambda^*/\Lambda \rightarrow \mathbb{Q}/\mathbb{Z}.$$

We call Λ^*/Λ with the map the *linking form* of Λ .

The lattice $\langle p \rangle$ denotes the lattice of rank 1 represented by the matrix $[p]$. The *standard negative definite lattice* of rank n is the lattice $\langle -1 \rangle^n$, the direct sum of n copies of $\langle -1 \rangle$. For a negative definite lattice Λ with rank n , we define d -invariant of Λ as

$$d(\Lambda) := \frac{n - \min_{\xi \in \text{Char}(\Lambda)} |\xi \cdot \xi|}{4}.$$

A negative definite lattice Λ can be uniquely decomposed as $\Lambda' \oplus \langle -1 \rangle^m$ for some m so that Λ' does not contain any vector with square -1 . Also, observe that $d(\Lambda) = d(\Lambda')$.

2.2. Restrictions on lattices bounded by a rational homology 3-sphere.

We recall well-known constraints on lattices bounded by a given rational homology 3-sphere. We also refer the readers to Owens and Strle's survey paper [OS05a] for more detail.

Linking pairings of 3-manifolds and linking forms of lattices. For a rational homology 3-sphere Y , the *linking pairing* of Y is the map

$$\lambda_Y : H^2(Y; \mathbb{Z}) \times H^2(Y; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

induced by the Poincaré duality of Y . Suppose Y bounds a lattice $\Lambda = (\mathbb{Z}^n, Q)$ and X is a 4-manifold realizing Λ . From the homology long exact sequence of the pair (X, Y) , we have

$$(1) \quad |H^2(Y; \mathbb{Z})| = |\det(\Lambda)|t^2$$

for some integer t , and it induces the following chain map

$$H^2(X, Y; \mathbb{Z})/Tors \xrightarrow{\mathcal{Q}} H^2(X; \mathbb{Z})/Tors \rightarrow H^2(Y; \mathbb{Z})/\mathcal{T},$$

where \mathcal{T} is the image of the torsion subgroup of $H^2(X; \mathbb{Z})$. This gives a necessary condition for a lattice to be bounded by Y .

Proposition 2.1. *If a lattice Λ is bounded by a rational homology 3-sphere Y , then there exist an integer t such that $|H^2(Y; \mathbb{Z})| = |\det(\Lambda)|t^2$, a subgroup \mathcal{T} of $H^2(Y)$ of order t , and a monomorphism*

$$\psi : \mathbb{Z}^n/Q(\mathbb{Z}^n) \rightarrow H^2(Y)/\mathcal{T}$$

preserving the linking form of $\mathbb{Z}^n/Q(\mathbb{Z}^n)$ and the induced linking pairing of $Im(\psi)$.

Indeed, for our purpose, we only need much weaker condition for lattices to be bounded by a rational homology 3-sphere as follow.

Proposition 2.2. *If a lattice Λ is bounded by a rational homology 3-sphere Y , then $\det(\Lambda)$ divides $|H^2(Y; \mathbb{Z})|$.*

Donaldson's Theorem. The celebrated Donaldson's diagonalization theorem can be used to give a constraint on definite lattices smoothly bounded by a 3-manifold. Recall the Donaldson's theorem.

Theorem 2.3 ([Don87, Theorem 1]). *Suppose X is a closed, smooth 4-manifold. If the intersection form of X is negative definite, then it is isometric to the standard definite lattice $(\mathbb{Z}^n, \langle -1 \rangle^n)$.*

Suppose a rational homology 3-sphere Y bounds a positive definite, smooth 4-manifold W . If Y bounds a negative definite lattice Λ , then we can construct a negative definite, closed, smooth 4-manifold by gluing a 4-manifold realizing Λ with W along Y . Then, by Donaldson's theorem, $\Lambda \oplus -Q_W$ embeds into $\langle -1 \rangle^{rk(\Lambda)+rk(Q_W)}$.

Example 2.4. The Poincaré homology sphere $-\Sigma$, oriented as the boundary of E_8 -plumbed 4-manifold, naturally bounds the E_8 lattice which is positive definite. It is well known that $-E_8$ lattice cannot be embedded into the standard negative definite lattice. Therefore, $-\Sigma$ cannot bound any negative definite smooth 4-manifold (including a 4-manifold with trivial intersection form), i.e. $\mathcal{I}(-\Sigma) = \emptyset$.

Heegaard Floer correction terms. Let Y be a rational homology 3-sphere and \mathfrak{t} be a spin^c structure over Y . In [OS03], Ozsváth and Szabó defined a rational valued invariant for (Y, \mathfrak{t}) called *the correction term* or *d-invariant*, denoted by $d(Y, \mathfrak{t})$. It is an analogous invariant to Frøyshov's in Seiberg-Witten theory [Fy96]. Among many important properties of the correction term, it gives a constraint on a definite 4-manifold bounded by Y .

Theorem 2.5 ([OS03, Theorem 9.6]). *If X is a negative definite, smooth 4-manifold bounded by Y , then for each spin^c structure \mathfrak{s} over X*

$$(2) \quad c_1(\mathfrak{s})^2 + n \leq 4d(Y, \mathfrak{s}|_Y),$$

where $c_1(-)$ denotes the first Chern class, n is the rank of $H_2(X; \mathbb{Z})$, and $\mathfrak{s}|_Y$ is the restriction of \mathfrak{s} over Y .

By analyzing the action of $H^2(X; \mathbb{Z})$ and $H^2(Y; \mathbb{Z})$ on the spin^c structures over X and Y respectively, we get the following necessary condition for a definite lattice bounded by Y . See [OS06, Theorem 2.2].

Proposition 2.6. *If a rational homology 3-sphere Y bounds a negative definite lattice Λ . Then $|H^2(Y; \mathbb{Z})| = |\det(\Lambda)|t^2$ for some integer $t \geq 0$, and there exists a subgroup \mathcal{T} of order t in $H^2(Y; \mathbb{Z})$, $x \in \text{Char}(\Lambda)$, $y \in \text{Spin}^c(Y)$ and a monomorphism*

$$\rho : \Lambda^* / \Lambda \rightarrow H^2(Y; \mathbb{Z}) / \mathcal{T}$$

such that

$$sq([\alpha]) + n \leq 4d_\rho([\alpha])$$

for any $[\alpha] \in \Lambda^* / \Lambda$, where

$$sq([\alpha]) := \max_{\beta \in [\alpha]} (x + 2\beta)^2$$

$$\text{and } d_\rho([\alpha]) := \min_{\beta \in \rho([\alpha])} d(Y, y + \beta).$$

Indeed, we only need the following weaker inequality in the later sections.

Proposition 2.7. *If a negative definite lattice Λ is smoothly bounded by a rational homology 3-sphere Y , then*

$$d(\Lambda) \leq d(Y),$$

where

$$d(Y) := \max_{\mathfrak{t} \in \text{Spin}^c(Y)} d(Y, \mathfrak{t}).$$

3. FINITENESS OF THE NUMBER OF DEFINITE LATTICES BOUNDED BY A RATIONAL HOMOLOGY 3-SPHERE

The purpose of this section is to prove Theorem 1.5 and consequently Theorem 1.3. First observe the following fact.

Lemma 3.1. *There are finitely many lattices which have a given rank and determinant.*

Proof. It is known that any lattice can be represented by a matrix, called the *reduced form*, of which the absolute values of entries are less than a certain function of the determinant and rank [Jon50, Chapter III]. Therefore, for a fixed rank and determinant, there are only finitely many reduced forms of definite lattices. \square

We prove the following algebraic lemma before showing Theorem 1.5.

Lemma 3.2. *For an odd prime p and odd integers s_1, s_2, \dots, s_{n-1} in $[-p+1, p-1]$, there exist an odd integer k and even integers k_1, k_2, \dots, k_{n-1} such that*

$$(3) \quad k^2 + \sum_{i=1}^{n-1} (ks_i - pk_i)^2 < \frac{n+2}{3}p^2.$$

Proof. Let p be an odd prime, s_1, s_2, \dots, s_{n-1} be odd integers in $[-p+1, p-1]$, and $K := \{-p+2, -p+4, \dots, p-2\}$. Observe that, for each $k \in K$ and s_i , there is a unique even integer k_i so that $|ks_i - pk_i| < p$. Denote this k_i by $k_i(k, s_i)$. Since p is an odd prime, $\{ks_i - p \cdot k_i(k, s_i) | k \in K\} = K$ for each $s_i \in K$. Therefore,

$$\begin{aligned} \sum_{k \in K} \sum_{i=1}^{n-1} (ks_i - p \cdot k_i(k, s_i))^2 &= (n-1) \cdot 2(1^2 + 3^2 + \dots + (p-2)^2) \\ &= \frac{n-1}{3}p(p-1)(p-2). \end{aligned}$$

Since $|K| = p-1$, there exists $k \in K$ such that

$$\sum_{i=1}^{n-1} (ks_i - p \cdot k_i(k, s_i))^2 \leq \frac{n-1}{3}p(p-2).$$

Since $|k| < p$, we obtain the desired inequality. \square

We first show a special case of Theorem 1.5 in which Γ is the empty lattice.

Proposition 3.3. *Let $C > 0$ and $D \in \mathbb{Z}$ be constants. There are finitely many negative definite lattices Λ , up to the stable-equivalence, which satisfy the following conditions:*

- $\det \Lambda = D$,
- $d(\Lambda) \leq C$, and
- Λ embeds into $\langle -1 \rangle^{rk(\Lambda)}$ with prime index.

Proof. Let Λ be a negative definite lattice of rank n that satisfies the conditions above. Without loss of generality, we may assume that there is no vector of square -1 in Λ . By Lemma 3.1 the theorem follows if we find an upper bound for the rank of Λ , only depending on D and C . Let $\{e, e_1, \dots, e_{n-1}\}$ be the standard basis of the standard negative definite lattice $\langle -1 \rangle^n$, i.e. $e^2 = -1$, $e \cdot e_i = 0$, $e_i \cdot e_j = -\delta_{ij}$ for $i, j = 1, \dots, n-1$.

Let p be the index of the embedding of Λ into $\langle -1 \rangle^n$. If $p = 1$, then the embedding is an isomorphism and so Λ is the empty lattice, up to the stable-equivalence. Now suppose p is an odd prime. Since the multiples of e give coset representatives of Λ in $\langle -1 \rangle^n$, we may write a basis of Λ , in terms of the standard basis of $\langle -1 \rangle^n$, as

$$\{pe, e_1 + s_1e, \dots, e_{n-1} + s_{n-1}e\}$$

for some odd integers s_i in $[-p+1, p-1]$. The matrix representation of Λ with respect to the basis is given as

$$Q = - \begin{pmatrix} p^2 & ps_1 & ps_2 & \cdots & ps_{n-1} \\ ps_1 & 1+s_1^2 & s_1s_2 & \cdots & s_1s_{n-1} \\ ps_2 & s_1s_2 & 1+s_2^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & s_{n-2}s_{n-1} \\ ps_{n-1} & s_1s_{n-1} & \cdots & s_{n-2}s_{n-1} & 1+s_{n-1}^2 \end{pmatrix}.$$

We also compute the inverse of Q as follows

$$Q^{-1} = \left(\begin{array}{c|cccc} -\frac{1+\sum_{i=1}^{n-1}s_i^2}{p^2} & \frac{s_1}{p} & \frac{s_2}{p} & \cdots & \frac{s_{n-1}}{p} \\ \hline \frac{s_1}{p} & & & & \\ \frac{s_2}{p} & & & & \\ \vdots & & & & \\ \frac{s_{n-1}}{p} & & & & \end{array} \right) \begin{array}{c} \\ \\ \\ -I_{(n-1) \times (n-1)} \\ \end{array}.$$

A characteristic covector ξ of Λ can be written as a vector

$$\xi = (k, k_1, \dots, k_{n-1}),$$

where k is an odd integer and k_i 's are even integers, in terms of the dual basis of Q . From the matrix Q^{-1} we compute

$$|\xi \cdot \xi| = \frac{1}{p^2} (k^2 + \sum_{i=1}^{n-1} (ks_i - pk_i)^2).$$

Applying Lemma 3.2,

$$\min\{|\xi \cdot \xi| : \xi \text{ characteristic covector of } L\} \leq \frac{n+2}{3}.$$

Therefore, by the condition

$$d(\Lambda) = \frac{n - \min_{\xi \in \text{Char}(\Lambda)} |\xi \cdot \xi|}{4} \leq C,$$

we have a bound for the rank of Λ , only depending on C . In fact the case that $p = 2$ is easier to find this bound by applying the same argument. Note that if $p = 2$, the inequality (3) is satisfied by choosing $k = k_1 = \dots = k_{n-1} = 0$. \square

Now, Theorem 1.5 will be proved by applying a similar argument of Theorem 3.3 to each embedding of nontrivial negative definite lattice Γ into the standard definite lattice. Let ι and ι' be embeddings of Γ into the standard negative definite lattices $\langle -1 \rangle^n$ and $\langle -1 \rangle^{n+k}$ respectively, $k \geq 0$. We say ι and ι' are *equivalent* if there is an embedding $j : \langle -1 \rangle^n \hookrightarrow \langle -1 \rangle^{n+k}$ such that $j \circ \iota = \iota'$.

Lemma 3.4. *Let Γ be a negative definite lattice. Then the number of embeddings of Γ into a standard negative definite lattice is finite, up to the equivalence above.*

Proof. Fix a basis $\{v_1, v_2, \dots, v_n\}$ for Γ . Note that an embedding of Γ into $\langle -1 \rangle^N$ can be presented by writing a basis of Γ in terms of the standard basis $\{e_1, e_2, \dots, e_N\}$, i.e.

$$v_i = \sum_{j=1}^N a_{i,j} e_j$$

for some $a_{i,j}$. We can show that there are finitely many choices of $a_{i,j}$ by finding them an explicit and inductive procedure along the rank of Γ .

Up to the automorphism of $\langle -1 \rangle^N$, we may assume that

$$v_1 = \sum_{j=1}^n a_{1,j} e_j$$

such that $a_{1,1} \geq a_{1,2} \geq \cdots \geq a_{1,n} \geq a_{1,n+1} = \cdots = a_{1,N} = 0$ and $\sum_{j=1}^n a_{1,j}^2 = v_1 \cdot v_1$. Note that the number of ways to describe each $|v_i \cdot v_i|$ as a sum of squares is finite. Thus there are finitely many embeddings of a lattice of rank 1 into $\langle -1 \rangle^N$.

Now, fix $a_{i,j}$ for $i < k$. Then v_k can be written as

$$v_k = \sum_{j=1}^N a_{k,j} e_j$$

for some $a_{k,j}$'s satisfying $\sum_{j=1}^N a_{k,j}^2 = v_k \cdot v_k$ and $\sum_{j=1}^N a_{l,j} a_{k,j} = v_l \cdot v_k$ for $l = 1, \dots, k-1$. Observe that such choices of $a_{k,j}$'s are finite up to the automorphisms of $\langle -1 \rangle^N$. \square

Proof of Theorem 1.5. Fix a negative definite lattice Γ , and constants $C > 0$ and $D \in \mathbb{Z}$. Let Λ be a negative definite lattice which satisfies the conditions in the theorem. Without loss of generality, we may assume that there is no square -1 vector in Λ . By Lemma 3.1, the theorem follows if we show that the rank of Λ is bounded by some constant only depending on Γ , C and D .

From the third condition of Λ , there is an embedding

$$\iota : \Gamma \hookrightarrow \langle -1 \rangle^{rk(\Lambda) + rk(\Gamma)}.$$

By Lemma 3.4 the number of such ι is finite, up to the automorphism of the standard definite lattice. Let us fix an embedding ι . Decompose as $(\text{Im } \iota)^\perp \cong \langle -1 \rangle^n \oplus E$, where E is a lattice without square -1 vectors. Observe that Λ embeds into $\langle -1 \rangle^n \oplus E$ and $rk(\Lambda) = n + rk(E)$.

Now we need to find a bound of the rank of Λ embedded in $\langle -1 \rangle^n \oplus E$. This will be obtained by the similar argument to Theorem 3.3. The main difference is that we have an extra summand E . However, this does not affect to show the bound of rank of Λ since the rank of E is also bounded by some function of Γ (Note that E is determined by the embedding of Γ). If $n = 0$ i.e. $\Lambda \hookrightarrow E$, then the rank of Λ is bounded by a function of Γ . Assume that $n \neq 0$.

Let p be the index of the embedding of Λ into $\langle -1 \rangle^n \oplus E$. If $p = 1$, then $\Lambda \cong (\text{Im } \iota)^\perp$, and hence $\Lambda \cong E$. Therefore, the rank of Λ is bounded by a function of Γ . Next suppose that p is an odd prime. Let

$$\{e, e_1, \dots, e_{n-1}, f_1, \dots, f_r\}$$

be a basis for $\text{Im}(\iota)^\perp \cong \langle -1 \rangle^n \oplus E$ satisfying $e^2 = -1$, $e \cdot e_i = 0$, $e_i \cdot e_j = -\delta_{ij}$ and $e \cdot f_j = e_i \cdot f_j = 0$ for any i, j and E is generated by $\{f_1, \dots, f_r\}$. We can choose a basis for Λ ,

$$(4) \quad \{pe, e_1 + s_1 e, \dots, e_{n-1} + s_{n-1} e, f_1 + t_1 e, \dots, f_r + t_r e\}$$

where s_i 's are odd integers in $[-p+1, p-1]$ and t_j 's are integers in $[-p+1, p-1]$. Now with respect to the Hom-dual coordinate for this basis, write a characteristic covector ξ as

$$\xi = (k, k_1, \dots, k_{n-1}, l_1, \dots, l_r)$$

where k is odd, k_i 's are even and $l_j \equiv (f_j + t_j e) \cdot (f_j + t_j e) \pmod{2}$ for each i, j . To find the matrices of Λ and Λ^{-1} , introduce an $(n+r) \times (n+r)$ matrix M and a $r \times r$ matrix A as

$$M_{ij} := \begin{cases} p & \text{if } i = 1, j = 1 \\ 1 & \text{if } i = j, 2 \leq j \leq n+r \\ s_{j-1} & \text{if } i = 1, 2 \leq j \leq n \\ t_{j-n} & \text{if } i = 1, n+1 \leq j \leq n+r \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{ij} := f_i \cdot f_j.$$

Note that M represents the embedding of Λ into $\langle -1 \rangle^n \oplus E$ and A represents E . By the basis in (4), Λ and the dual of Λ are represented as follows:

$$Q_\Lambda = M^t \begin{pmatrix} -I_{n \times n} & 0 \\ 0 & A \end{pmatrix} M$$

and

$$\begin{aligned} Q_\Lambda^{-1} &= M^{-1} \begin{pmatrix} -I_{n \times n} & 0 \\ 0 & A^{-1} \end{pmatrix} (M^t)^{-1} \\ &= -M^{-1} (M^t)^{-1} + M^{-1} \begin{pmatrix} 0 & 0 \\ 0 & A^{-1} + I_{r \times r} \end{pmatrix} (M^t)^{-1}. \end{aligned}$$

From Q_Λ^{-1} , we obtain that

$$|\xi \cdot \xi| = \frac{1}{p^2} (k^2 + \sum_{i=1}^{n-1} (ks_i - pk_i)^2) + S(k, l_1, \dots, l_r, t_1, \dots, t_r),$$

for some function S . Note that the function S is independent to n . Then by Lemma 3.2,

$$\min\{|\xi \cdot \xi| : \xi \text{ characteristic covector of } L\} \leq \frac{n}{3} + A$$

for some constant $A(\Gamma, D)$. Therefore, since $d(\Lambda) \leq C$, there is a bound on n , and hence the rank of Λ , only depending on Γ , C and D . In fact the case that $p = 2$ is easier to find this bound by applying the same argument.

For an arbitrary index p , we use an idea in [OS12a] to have a sequence of embeddings

$$\Lambda = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \dots \hookrightarrow E_s = (\text{Im} \iota)^\perp$$

such that each embedding $E_i \hookrightarrow E_{i+1}$ has a prime index. The length of this steps is also bounded by some constant related to D . Moreover, $d(E_i) \leq d(\Lambda) \leq C$ for any i since $E_i^* \hookrightarrow \Lambda^*$ and so $\text{Char}(E_i) \subset \text{Char}(\Lambda)$. Thus we complete the proof of theorem by an induction along each prime index embedding. \square

Proof of Theorem 1.3. Let Y be a rational homology 3-sphere that bounds a positive definite, smooth 4-manifold W . By the necessary conditions for a negative definite lattice to be bounded by Y discussed in Section 2.2, $\mathcal{I}(Y)$ is a subset of the union of

$$\mathcal{L}(-Q_W, \max_{\mathfrak{t} \in \text{Spin}^c(Y)} d(Y, \mathfrak{t}), D)$$

over all integers D dividing $|H_1(Y; \mathbb{Z})|$. Then Theorem 1.3 follows from Theorem 1.5. \square

APPENDIX A. SEIFERT FIBERED RATIONAL HOMOLOGY 3-SPHERES THAT BOUND DEFINITE 4-MANIFOLDS OF BOTH SIGNS

In this appendix we discuss which Seifert fibered 3-manifolds can bound both positive and negative definite smooth 4-manifolds.

A.1. Seifert fibered spaces. Seifert fibered 3-manifolds are a large class of 3-manifolds that contains 6 geometries among Thurston's 8 geometries of 3-manifolds. A Seifert fibered rational homology 3-sphere can be represented by a Seifert form

$$M(e_0; (a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)),$$

where e_0 , a_i 's and b_i 's are integers and $\gcd(a_i, b_i) = 1$, and Dehn-surgery diagram of the corresponding 3-manifold is depicted in Figure A.1 for the case that $k = 3$. A Seifert fibered 3-manifold Y of a form $M(e_0; (a_1, b_1), \dots, (a_k, b_k))$ naturally bounds a 4-manifold constructed by the plumbing diagram in Figure A.1, in which $\alpha_{ij} \in \mathbb{Z}$ are determined by the following Hirzebruch-Jung continued fraction:

$$\frac{a_i}{b_i} = [\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ik_i}] = \alpha_{i1} - \frac{1}{\alpha_{i2} - \frac{1}{\ddots - \frac{1}{\alpha_{ik_i}}}},$$

where $\alpha_{ij} \geq 2$ for $1 \leq i \leq k$ and $2 \leq j \leq k_i$.

By Rolfsen's twist, it is easy to see that $M(e_0; (a_1, b_1), (a_2, b_2), \dots, (a_k, b_k))$ and $M(e_0 + n; (a_1, b_1), \dots, (a_j, b_j - na_j), \dots, (a_k, b_k))$ represent the same 3-manifold: see for example [GS99, Chapter 5.3]. Hence any Seifert fibered rational homology 3-sphere admits a canonical Seifert form, $M(e_0; (a_1, b_1), (a_2, b_2), \dots, (a_k, b_k))$ such that $a_i > b_i > 0$ for all $1 \leq i \leq k$. We refer the form by the *normal form* of a Seifert fibered rational homology 3-sphere.

One can decide negative definiteness of the intersection form of the 4-manifold induced from a normal form by the following lemma.

Lemma A.1. *The intersection form of the corresponding plumbed 4-manifold of a normal form $M(e_0; (a_1, b_1), \dots, (a_k, b_k))$ is negative definite if and only if*

$$e(M) := e_0 + \frac{b_1}{a_1} + \dots + \frac{b_k}{a_k} < 0.$$

We refer $e(M)$ by the *Euler number* of a Seifert form M . Observe that $e(-M) = -e(M)$.

Proposition A.2. *Any Seifert fibered rational homology 3-sphere can bound either positive or negative definite smooth 4-manifolds.*

Proof. Let Y be a Seifert fibered rational homology 3-sphere with the normal form M . Since Y is a rational homology 3-sphere, $e(M) \neq 0$. If $e(M) < 0$, then Y bounds a negative definite 4-manifold. If $e(M) > 0$, then $-Y$ bounds a negative definite 4-manifold since $e(-M) = -e(M) < 0$. Thus Y bounds a positive definite one. \square

Now we give a condition for a Seifert fibered rational homology 3-sphere to bound definite 4-manifolds of both signs.

Proposition A.3. *Let Y be a Seifert fibered rational homology 3-sphere of the normal form*

$$M(e_0; (a_1, b_2), \dots, (a_k, b_k)).$$

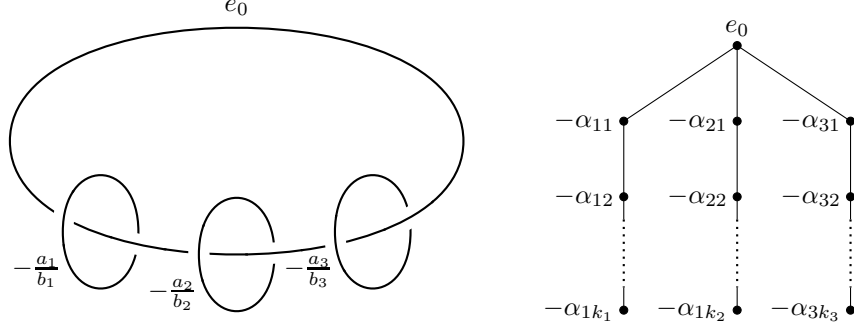


FIGURE A.1. The surgery diagram and plumbing diagram of Seifert manifold $M(e_0; (a_1, b_1), (a_2, b_2), (a_3, b_3))$.

If $e_0 + k \leq 0$, then Y bounds both positive and negative definite smooth 4-manifolds.

Proof. By the previous proposition, it is easy to see that Y bounds a negative definite 4-manifold if $e_0 + k \leq 0$. To find a positive bounding of Y , consider a plumbed 4-manifold X corresponding to $-Y \cong M(-e_0 - k; (a_1, a_1 - b_1), (a_2, a_2 - b_2), \dots, (a_k, a_k - b_k))$. It is easy to check that $b_2^+(X) = 1$. By $(e_0 + k)$ blowing-ups of \mathbb{CP}^2 on the sphere corresponding to the central vertex in X , we get a self intersection 0-sphere in $X \# (\overline{\mathbb{CP}^2})^{e_0+k}$. By doing a surgery on this sphere, we obtain a desired negative definite 4-manifold. More precisely, we remove the interior of the tubular neighborhood of the sphere, $S^2 \times D^2 \subset X \# (\overline{\mathbb{CP}^2})^{e_0+k}$, and glue $D^3 \times S^1$ along the boundary, and it reduces $b_2^+(X \# (\overline{\mathbb{CP}^2})^{e_0+k})$ by 1. \square

Remark. This proposition also follows since these Seifert fibered spaces can be obtained by the branched double covers of S^3 along alternating Montesinos links. See for example [MO07, Section 4].

Note that the condition in Proposition A.3 is not a necessary condition. For example, the Brieskorn manifold, $\Sigma(2, 3, 6n + 1) \cong M(-1, (2, 1), (3, 1), (6n + 1, 1))$ bounds both negative and positive definite 4-manifolds since $e(M) < 0$ and it can be obtained by $(+1)$ -surgery of S^3 on the n -twist knot.

On the other hand, the inequality $e_0 + k \leq 0$ is sharp since the Brieskorn manifold $\Sigma(2, 3, 5) \cong M(-2, (2, 1), (3, 2), (5, 4))$, of which $e_0 + k = 1$, cannot bound any positive definite 4-manifold by the constraint from Donaldson's diagonalization theorem. Note that $\Sigma(2, 3, 5)$ is the Poincaré homology sphere Σ in Example 2.4.

A.2. Links of quotient surface singularity. One interesting class of Seifert fibered 3-manifolds is the links of quotient surface singularities. They are all small Seifert fibered spaces and the classification of symplectic fillings of them is well understood recently. See Bhupal and Ono [BO12] for example. In this section we completely determine whether a link of quotient surface singularity can admit smooth definite fillings of both signs. We use the notations in Bhupal and Ono's paper [BO12] for this class of 3-manifolds. We assume that a given link of quotient surface singularity is oriented by the natural complex structure.

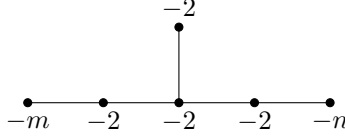


FIGURE A.2. A configuration of which the corresponding lattice cannot be embedded into the standard definite lattice.

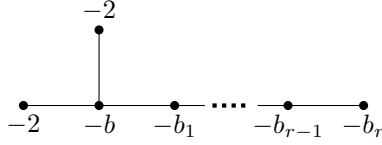


FIGURE A.3. The plumbing graph of the minimal resolution of $D_{n,q}$, where $\frac{n}{q} = [b, b_1, \dots, b_{r-1}, b_r]$.

Proposition A.4. *Any links of surface quotient singularities except T_1 , O_1 , I_1 and I_7 can bound both positive and negative definite smooth 4-manifolds. The links of T_1 , O_1 , I_1 and I_7 cannot bound a positive definite smooth 4-manifolds.*

Proof. Let Y be a link of surface quotient singularity. If Y is either T_1 , O_1 , I_1 or I_7 , we can use an argument by Lecuona and Lisca in [LL11] to show that each of these 3-manifolds never bound a positive definite 4-manifold. They gave the configuration in Figure A.2 that cannot be embedded into the standard definite lattice, and indeed these 4-cases contain the configuration.

Since each link of singularities bounds a natural negative definite 4-manifold (the minimal resolution), it is enough to construct a positive definite 4-manifold with the given boundary Y except the 4-cases. In the cyclic quotient cases, the links are lens spaces, so these bound positive and negative definite plumbed 4-manifolds.

In dihedral case $D_{n,q}$, where $1 < q < n$ and $(n, q) = 1$, the minimal resolution graph of a link $Y(n, q)$ is given in Figure A.3. Let $\frac{n}{q} = [b, b_1, \dots, b_{r-1}, b_r]$, and note that $b \geq 2$ since $1 < q < n$. If $b > 2$, there is a positive definite bounding by Proposition A.3. In the case $b = 2$, we can check that $-Y(n, q) \cong M(-1; (2, 1), (2, 1), (q, bq - n))$, and the corresponding plumbed 4-manifold X satisfies $b_2^+(X) = 1$. As seen in Figure A.4, we get a 0-framed 2-sphere after blowing down twice from X , and we obtain a desired 4-manifold by a surgery along the sphere. It is easy to check that this resulting 4-manifold is negative definite. In the other cases (tetrahedral, octahedral and icosahedral cases), we can apply similar argument except the 4-cases. □

As we mentioned in the introduction, it is known that the 3-manifolds obtained by the double branched cover of a quasi-alternating link in S^3 bound both positive and negative definite 4-manifolds with trivial H_1 . Indeed, there are some family of Seifert fibered 3-manifolds that are not obtained by double branched cover on a quasi-alternating link but can be shown to bound definite 4-manifolds of both signs by our result. For example, the links of $D_{n,n-1}$ -singularities are such 3-manifolds. We use an obstruction of Greene [Gre13] to show the following.

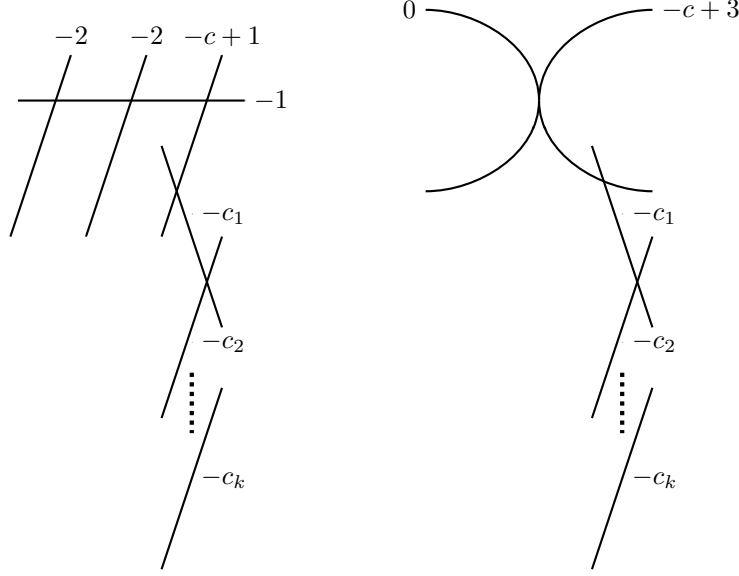


FIGURE A.4. The compactifying divisor of $D_{n,q}$ and the configuration after blow-down twice, where $\frac{n}{n-q} = [c, c_1, \dots, c_{k-1}, c_k]$.

Proposition A.5. *The link of $D_{n,n-1}$ -singularities cannot bound a positive definite smooth 4-manifold with trivial H_1 , and consequently cannot be obtained by the double branched cover of S^3 along a quasi-alternating link in S^3 .*

Proof. Let Y be the link of $D_{n,n-1}$ -singularity, and X be the minimal resolution of Y . Suppose W is a positive definite 4-manifold bounded by Y . Then, as usual, $Q_X \oplus -Q_W$ embeds into $\langle -1 \rangle^{rk(Q_X) + rk(Q_W)}$. First observe that an embedding ι of Q_X to a standard definite lattice is unique, up to the automorphism of the standard lattice, as depicted in Figure A.5, in terms of the standard basis $\{e_1, e_2, \dots, e_{n+1}\}$ of the standard lattice. For this embedding, we have $(\text{Im} \iota)^\perp \cong \langle -1 \rangle^{rk(Q_W)}$.

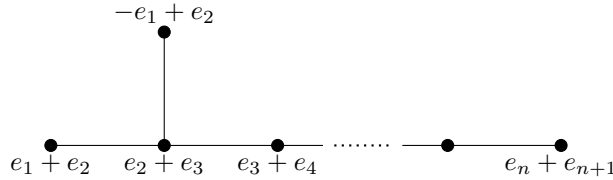


FIGURE A.5. The minimal resolution of $D_{n,n-1}$ -singularity

Since $H_1(X; \mathbb{Z})$ is trivial, $-Q_W$ is isomorphic to $(\text{Im} \iota)^\perp \cong \langle -1 \rangle^{rk(Q_W)}$. Suppose $H_1(W)$ is trivial. Then $H^2(W, \partial W)$ and $H^2(W)$ are torsion free, and $H^2(Y)$ have to be trivial from the following long exact sequence:

$$\cdots \rightarrow H^2(W, \partial W) \xrightarrow[\cong]{Q_W} H^2(W) \rightarrow H^2(Y) \rightarrow H^3(W, \partial W) = 0$$

However, we know that $H^2(Y)$ is non-trivial. □

Remark. Recently, all quasi-alternating Montesinos links are completely classified due to Issa in [Iss17]. Since any Seifert fibered rational homology 3-sphere is the double branched cover of S^3 along a Montesinos link, the above proposition might be followed from his result.

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